关于叶片的等距对应关系

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摘要. 我们证明，对于一个典型的 3 维可积滚动分布（排除可展种子和各向同性可展叶）而言，具有普遍性质的叶的等距对应（独立于种子的形状）需要 Bäcklund 变换。

目录

1. 介绍

我们将考虑实 3 维欧几里得空间的复化

$$(\mathbb{C}^3, \langle\cdot,\cdot\rangle),\ \langle x, y\rangle := x^T y,\ |x|^2 := x^T x,\ x, y \in \mathbb{C}^3$$

；在此设置下，曲面是 $\mathbb{C}^3$ 中的 2 维对象，依赖于两个实或复参数。

各向同性 (null) 向量是长度为 0 的向量；由于大多数向量不是各向同性的，我们称一个向量为简单的向量，并将强调各向同性以区分各向同性向量。同样的命名将在其他情况下适用：例如，我们将非退化二次曲面称为二次曲面（与复单位球面射影等价的二次曲面）。

考虑 Lie 的观点：可以用一个 $x \subset \mathbb{C}^3$ 的表面替换为一个 2 维的*接触单元*（点和通过这些点的平面的对；经典几何学家称它们为*面 facet*）分布：它的切平面集合（突出显示了切点）；因此，接触元素是表面的无穷小版本（表面的整体元素 $(x, dx)|_{\mathrm{pt}}$）。反之，一个 2 维的接触元素分布并不总是表示一个表面的切平面集合（突出显示了切点），但当一个 2 维的接触元素分布是可积的时候（即它是叶状子流形的切平面集合）并不能区分这个子流形是一个曲面、曲线还是点，因此允许叶状子流形退化。

一个 3 维的接触元素分布是可积的，如果它是 1 维叶状集合的切平面集。

两个*可卷曲的*（适用或等距）表面可以*滚动*（应用）到彼此，使得在任何时刻它们相切并且在切点处具有相同的微分。

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定义 1.1. 两个等距曲面 x_0, $x \subset \mathbb{C}^3$（即 $|dx_0|^2 = |dx|^2$）的滚动是这样一个曲面、曲线或点 $(R, t) \subset \boldsymbol{O}_3(\mathbb{C}) \ltimes \mathbb{C}^3$，使得 $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$。

滚动引入了平联络形式（它编码了第二基本形式的差异 x_0，x，而它是平的则编码了 Gauß-Codazzi-Mainardi-Peterson 方程的差异 x_0，x）。

定义 1.2. 考虑一个可积的 3 维接触元素分布 $\mathcal{F} = (p, m)$，其中心位于 $p = p(u, v, w)$，法向场为 $m = m(u, v, w)$，并且沿表面 $x_0 = x_0(u, v)$ 分布。如果我们在一个等距表面 x 上滚动 x_0（即 $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$），则接触元素的分布为 $(Rp + t, Rm)$，并沿着 x 分布；如果对于任何滚动都保持可积，则该分布被称为具有种子 x 和叶 $Rp + t$ 的 3-维可积滚动分布的接触元素。

Bianchi 考虑了 Bäcklund 变换的最一般形式作为一对焦点曲面（一个曲面是另一个曲面的变换）的维尔廷格讷特一致集（其两个焦点面上的渐近方向对应；等价地说，它们的第二基本形式成比例）。请注意，尽管 Weingarten 一致集提供的对应关系不给出适用性（等距）对应，Bäcklund 变换是通过几何变换解决等距变形问题的最佳工具，因为它提供了等距变形特征的对应关系（根据 Darboux 的说法，这些特征是渐近方向），它直接关联到无穷小等距变形问题（Darboux 证明了无穷小等距变形生成 Weingarten 一致集，而 Guichard 证明了相反情况：Weingarten 一致集中一个焦点面的方向正常于另一个焦点面存在无穷小等距变形；参见 Darboux ([2],§883-§)），并且它允许其第二次迭代的 Bianchi 可交换性定理版本。

当 $\tilde{d}\cdot := \partial_u \cdot du + \partial_v \cdot dv + \partial_w \cdot dw = d \cdot + \partial_w \cdot dw, V := p - x_0$ 如果接触元素的 3 维分布是可积的，并且当我们滚动 x_0 到一个等距表面 x，$(x, dx) = (R, t)(x_0, dx_0)$（即我们用 x，RV，Rm 替换 x_0，V，m）时，卷动分布仍然保持可积，则沿着叶状区域我们有

$$0 = (Rm)^T \tilde{d}(RV + x) = m^T(\omega \times V + d(V + x_0) + \partial_w V dw),$$
$$\omega := N_0 \times R^{-1} dR N_0,\ N_0 := \frac{\partial_u x_0 \times \partial_v x_0}{|\partial_u x_0 \times \partial_v x_0|}. \tag{1.1}$$

由于我们不需要这个一般可积滚动分布的接触元素的可积性条件，因此我们将不推导它。

定义 1.3. 一个 3 维可积的滚动分布称为典型的如果 $m^T \partial_w V(m \times V) \times N_0 \neq 0$。

在 [3] 中我们证明了对于一个典型的 3 维可积滚动分布的接触元素（排除可展种子和各向同性可展叶）以及切触配置的对称性（接触元素集中在表面 x_0 的切平面上，并且进一步通过这些切平面的原点），种子和任何叶片都是 Weingarten 符合的焦面（因此我们得到了根据 Bianchi 定义的 Bäcklund 变换）并且对于一个典型的 3 维可积滚动分布的接触元素（排除可展种子和各向同性可展叶），叶片之间的等距对应具有普遍性质（独立于种子的形状）需要切触配置（接触元素集中在表面 x_0 的切平面上）。进一步应用变量变换 $w = w(\tilde{w}, u, v)$，我们得到 $N_0^T[d(V + x_0) \times \wedge d(V + x_0)] \neq 0$ 并获得条件

$$d(V + x_0)^T (I_3 - \frac{m N_0^T}{m^T N_0}) \odot [d(\frac{N_0 m^T}{m^T N_0}) \partial_w V - \partial_w (\frac{N_0 m^T}{m^T N_0}) d(V + x_0) +$$
$$2 N_0^T [\partial_w V \times d(V + x_0)] \frac{d(\frac{N_0 m^T}{m^T N_0}) \wedge d(V + x_0)}{N_0^T [d(V + x_0) \times \wedge d(V + x_0)]}] = 0,\ du \wedge dv \wedge dw \neq 0. \tag{1.2}$$

根据 Bianchi[1]（指代 3 维可积滚动分布的接触元素）但是，考虑到最终应用于变形问题的应用，有必要将问题限定得更多，并假设每个面片 f 及其相关的每个面片 f' 的中心都在另一个的平面上。。

我们现在有了本文的主要**定理**：

定理 1.4. *对于一个通用的 3 维可积滚动分布（排除可展种子和各向同性可展叶）而言，一般性质的叶之间的等距对应（独立于种子的形状）需要 Bäcklund 变换。*

论文的剩余部分组织如下：在第 2 节中我们回顾了曲面的滚动问题，并在第 3 节中给出了**定理** 1.4 的证明。

2. 表面的滚动问题

令 $(u,v)\in D$ 在 D 的定义域为 $\mathbb{R}^2$ 或 $\mathbb{C}^2$，且 $x:D\mapsto\mathbb{C}^3$ 是一个曲面。

对于在 D 和 $a,b\in\mathbb{C}^3$ 上的 ω_1,ω_2 $\mathbb{C}^3$ 值 1 形式，我们有

$$a^T\omega_1\wedge b^T\omega_2=((a\times b)\times\omega_1+b^T\omega_1 a)^T\wedge\omega_2=(a\times b)^T(\omega_1\times\wedge\omega_2)+b^T\omega_1\wedge a^T\omega_2;$$

$$\text{in particular } a^T\omega\wedge b^T\omega=\frac{1}{2}(a\times b)^T(\omega\times\wedge\omega). \tag{2.1}$$

由于 $\times$ 和 $\wedge$ 都是斜对称的，我们有 $2\omega_1\times\wedge\omega_2=\omega_1\times\omega_2+\omega_2\times\omega_1=2\omega_2\times\wedge\omega_1$。

考虑标量积 $\langle\cdot,\cdot\rangle$ 在 $\boldsymbol{M}_3(\mathbb{C})$：$\langle X,Y\rangle:=\frac{1}{2}\mathrm{tr}(X^TY)$ 上。我们有等距变换

$$\alpha:\mathbb{C}^3\mapsto\boldsymbol{o}_3(\mathbb{C}),\ \alpha\left(\begin{bmatrix}x^1\\x^2\\x^3\end{bmatrix}\right)=\begin{bmatrix}0&-x^3&x^2\\x^3&0&-x^1\\-x^2&x^1&0\end{bmatrix},\ x^Ty=\langle\alpha(x),\alpha(y)\rangle=\frac{1}{2}\mathrm{tr}(\alpha(x)^T\alpha(y)),$$

$$\alpha(x\times y)=[\alpha(x),\alpha(y)]=\alpha(\alpha(x)y)=yx^T-xy^T,\ \alpha(Rx)=R\alpha(x)R^{-1},\ x,y\in\mathbb{C}^3,\ R\in\boldsymbol{O}_3(\mathbb{C}).$$

令 $x\subset\mathbb{C}^3$ 是一个可以与表面 $x_0\subset\mathbb{C}^3$ 应用（等距）的表面：

$$(x,dx)=(R,t)(x_0,dx_0):=(Rx_0+t,Rdx_0), \tag{2.2}$$

其中 (R,t) 是 $\boldsymbol{O}_3(\mathbb{C})\ltimes\mathbb{C}^3$ 中的一个子流形（通常是一个曲面，但如果 x_0，x 是直纹面且在等距下对应的直线族，则为一条曲线，或者如果 x_0，x 仅相差一个刚体运动则为一个点）。子流形 R 给出了 x_0 在 x 上的滚动，也就是说，如果我们刚性地将 x_0 滚动在 x 上，使得等距下的对应点具有相同的微分，R 将决定 x_0 的旋转；平移 t 将满足 $dt=-dRx_0$。

对于 (u,v) 在 x_0，x 上的参数化以及在 x_0，x 的各向异性（退化）诱导度量轨迹之外，我们有 $N_0:=\frac{\partial_ux_0\times\partial_vx_0}{|\partial_ux_0\times\partial_vx_0|}$，$N:=\frac{\partial_ux\times\partial_vx}{|\partial_ux\times\partial_vx|}$，分别是有正向朝向的单位法向量场 x_0，x 和 R 由 $R=[\partial_ux\ \ \partial_vx\ \ N][\partial_ux_0\ \ \partial_vx_0\ \ \det(R)N_0]^{-1}$ 确定；我们取 R 与 $\det(R)=1$；因此，滚动与 x_0（或 x 的另一面）的另一面的旋转是 $R':=R(I_3-2N_0N_0^T)=(I_3-2NN^T)R,\ \det(R')=-1$。

因此，$\boldsymbol{O}_3(\mathbb{C})\ltimes\mathbb{C}^3$ 作用于 2 维可积分布的接触元素 (x_0,dx_0) 在 $T^*(\mathbb{C}^3)$ 中如下：$(R,t)(x_0,dx_0)=(Rx_0+t,Rdx_0)$；滚动是一个子流形 $(R,t)\subset\boldsymbol{O}_3(\mathbb{C})\ltimes\mathbb{C}^3$，使得 $(R,t)(x_0,dx_0)$ 仍然是可积的。

我们有：

$$R^{-1}dRN_0=R^{-1}dN-dN_0. \tag{2.3}$$

应用兼容性条件 d 到 (2.2) 我们得到:

$$R^{-1}dR \wedge dx_0 = 0,\ dRR^{-1} \wedge dx = 0. \tag{2.4}$$

由于 $R^{-1}dR$ 是斜对称的，并且使用 (2.4)，我们得到

$$dx_0^T R^{-1}dRdx_0 = 0. \tag{2.5}$$

从 (2.5) 对于 $a \in \mathbb{C}^3$ 我们得到 $R^{-1}dRa = R^{-1}dR(a^\perp + a^\top) = a^T N_0 R^{-1}dRN_0 - a^T R^{-1}dRN_0N_0 = \omega \times a,\ \omega := N_0 \times R^{-1}dRN_0 =^{(2.3)} (\det R)R^{-1}(N \times dN) - N_0 \times dN_0 = R^{-1}(N \times dN) - N_0 \times dN_0$。因此，$R^{-1}dR = \alpha(\omega)$ 和 ω 是在 T^*x_0 中的平坦联络形式:

$$d\omega + \frac{1}{2}\omega \times \wedge\omega = 0,\ \omega \times \wedge dx_0 = 0,\ (\omega)^\perp = 0. \tag{2.6}$$

通过 $s := N_0^T(\omega \times dx_0) = s_{11}du^2 + s_{12}dudv + s_{21}dvdu + s_{22}dv^2$ 作为 x, x_0 的第二基本形式之差，我们有

$$\omega = \frac{s_{12}\partial_u x_0 - s_{11}\partial_v x_0}{|\partial_u x_0 \times \partial_v x_0|}du + \frac{s_{22}\partial_u x_0 - s_{21}\partial_v x_0}{|\partial_u x_0 \times \partial_v x_0|}dv; \tag{2.7}$$

($\omega \times \wedge dx_0 = 0$ 等价于 $s_{12} = s_{21}$; $(d\omega)^\perp + \frac{1}{2}\omega \times \wedge\omega = 0$, $(d\omega)^\top = 0$ 分别编码了 Gauß-Codazzi-Mainardi-Peterson 方程的差值 x_0 和 x)。

使用 $\frac{1}{2}dN_0 \times \wedge dN_0 = K|\partial_u x_0 \times \partial_v x_0|N_0 du \wedge dv$, K 作为 Gauß 曲率，我们得到 $dN_0 \times \wedge dN_0 = R^{-1}(dN \times \wedge dN) =^{(2.3)} (\omega \times N_0 + dN_0) \times \wedge(\omega \times N_0 + dN_0) = dN_0 \times \wedge dN_0 + 2(\omega \times N_0) \times \wedge dN_0 + \omega \times \wedge\omega$; 因此

$$\frac{1}{2}\omega \times \wedge\omega = dN_0^T \wedge \omega N_0. \tag{2.8}$$

请注意

$$\omega' = N_0 \times R'^{-1}dR'N_0 = -\omega - 2N_0 \times dN_0 \tag{2.9}$$

和

$$a^T \wedge \omega = 0,\ \forall\omega \text{ satisfying (2.6) for } a\ 1-\text{form} \Rightarrow\ a^T \odot dx_0 := \frac{a^T dx_0 + dx_0^T a}{2} = 0. \tag{2.10}$$

请注意，反向 $a^T \odot dx_0^0 = 0$, a $1-$ 形式 $\Rightarrow a^T \wedge \omega = 0, \forall\omega$ 满足 (2.6) 也是成立的。

3. 定理的证明 1.4

3.1. 情况 $m = V \times N_0 + \boldsymbol{m}N_0 + \boldsymbol{n}V$.

我们有 $0 = (Rm)^T\tilde{d}(RV + x) = m^T(\omega \times V + d(V + x_0) + \partial_w V dw)$，或者假设 $m^T\partial_w V \neq 0$，

$$dw = \frac{N_0^T[V \times d(V + x_0)] + \boldsymbol{m}V^T(\omega \times N_0 + dN_0) - \boldsymbol{n}V^T d(V + x_0)}{N_0^T(\partial_w V \times V) + \boldsymbol{n}V^T\partial_w V}.$$

将兼容性条件 d 应用于此方程并使用该方程本身，我们得到

$$0 = -\frac{\partial_w[N_0^T[V \times d(V + x_0)] + \boldsymbol{m}V^T(\omega \times N_0 + dN_0) - \boldsymbol{n}V^T d(V + x_0)]}{N_0^T(\partial_w V \times V) + \boldsymbol{n}V^T\partial_w V}\wedge$$
$$\frac{N_0^T[V \times d(V + x_0)] + \boldsymbol{m}V^T(\omega \times N_0 + dN_0) - \boldsymbol{n}V^T d(V + x_0)}{N_0^T(\partial_w V \times V) + \boldsymbol{n}V^T\partial_w V} +$$

$$
d\frac{N_0^T[V\times d(V+x_0)]}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}+d\frac{\boldsymbol{m}V^T}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}\wedge(\omega\times N_0+dN_0)-
$$

$$
d\frac{\boldsymbol{n}V^T}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}\wedge d(V+x_0)
$$

$$
\overset{(2.1)}{=}-\frac{\partial_w[N_0^T[V\times d(V+x_0)]-\boldsymbol{n}V^Td(V+x_0)]}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}\wedge\frac{N_0^T[V\times d(V+x_0)]-\boldsymbol{n}V^Td(V+x_0)}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}+
$$

$$
d\frac{N_0^T[V\times d(V+x_0)]}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}-d\frac{\boldsymbol{n}V^T}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}\wedge d(V+x_0)-
$$

$$
\frac{\boldsymbol{m}^2N_0^T(\partial_w V\times V)N_0^T(dN_0\times\wedge dN_0)}{2[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}+[\frac{\partial_w\boldsymbol{m}[N_0^T[V\times d(V+x_0)]-\boldsymbol{n}V^Td(V+x_0)]}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}-
$$

$$
\frac{\boldsymbol{m}\partial_w[N_0^T[V\times d(V+x_0)]-\boldsymbol{n}V^Td(V+x_0)]}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}+d\frac{\boldsymbol{m}}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}-
$$

$$
\frac{\boldsymbol{m}(\partial_w V\times N_0)^Td(V+x_0)}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}]\wedge V^T(\omega\times N_0+dN_0)+
$$

$$
\frac{\boldsymbol{mn}N_0^T[\partial_w V\times d(V+x_0)]}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}\wedge(N_0\times V)^T(\omega\times N_0+dN_0)
$$

$$
\overset{(2.1)}{=}\frac{1}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}[-|V|^2d\boldsymbol{n}\wedge(N_0\times\partial_w V)^Td(V+x_0)-\frac{1}{2}[N_0^T(\partial_w V\times V)(1+\boldsymbol{n}^2)
$$

$$
-\partial_w\boldsymbol{n}|V|^2]N_0^T[d(V+x_0)\times\wedge d(V+x_0)]-\frac{1}{2}|V|^2[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]N_0^T(dN_0\times\wedge dN_0)
$$

$$
+[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]N_0^T[dV\times\wedge d(V+x_0)]-(N_0\times\partial_w V+\boldsymbol{n}\partial_w V)^TdV\wedge(N_0\times V)^Td(V+x_0)
$$

$$
+\boldsymbol{n}[(dV\times V)\times(N_0\times\partial_w V+\boldsymbol{n}\partial_w V)]^T\wedge d(V+x_0)-\frac{\boldsymbol{m}^2}{2}N_0^T(\partial_w V\times V)N_0^T(dN_0\times\wedge dN_0)]
$$

$$
+\frac{1}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}[d\boldsymbol{m}+\frac{\partial_w\boldsymbol{m}[N_0^T[V\times d(V+x_0)]-\boldsymbol{n}V^Td(V+x_0)]}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}
$$

$$
+\frac{\boldsymbol{m}[\partial_w\boldsymbol{n}V^Td(V+x_0)+\boldsymbol{n}\partial_w V^Td(V+x_0)]}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}-\frac{\boldsymbol{m}[N_0^T(\partial_w V\times dV)+d\boldsymbol{n}V^T\partial_w V+\boldsymbol{n}dV^T\partial_w V]}{N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V}]\wedge
$$

$$
V^T(\omega\times N_0+dN_0)+\frac{\boldsymbol{mn}N_0^T[\partial_w V\times d(V+x_0)]}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}\wedge(N_0\times V)^T(\omega\times N_0+dN_0).
$$

这可以简写为

$$
(3.1)\; A+B\wedge V^T(\omega\times N_0+dN_0)+C\wedge(N_0\times V)^T(\omega\times N_0+dN_0)=0,\ \forall\,\omega \text{ satisfying } (2.6),
$$

其中 A 是一个标量 2 形式不依赖于 ω 和 B, C 是标量 1 形式不依赖于 ω

为了使 w 能够由 ω 根据 (1.1) 确定，我们需要 $(m\times V)\times N_0\neq 0$ 和 w 不能通过任何其他关系（无论是函数关系（如 (3.1) 先验地是）还是微分关系）与 ω 相关联，因此在 (3.1) 中，ω 独立于 w 消去，并且在外边 w 我们可以用任何其他 (2.6) 的解来替换 ω。

用 $0, -2N_0\times dN_0, -\omega-2N_0\times dN_0$ 分别替换 ω，我们得到 $A=0$ 和

$$
(3.2)\quad (BV+CN_0\times V)^T\wedge dN_0=0,\ (BN_0\times V-CV)^T\wedge\omega=0,\ \forall\,\omega \text{ satisfying } (2.6).
$$

从 (2.10) 我们根据 (3.2) 的第二个关系式得到 $(BN_0\times V-CV)^T\odot dx_0=0$；结合 $B=B_udu+B_vdv$，$C=C_udu+C_vdv$ 这就变成了 $B_u(N_0\times V)^T\partial_ux_0-C_uV^T\partial_ux_0=B_v(N_0\times V)^T\partial_vx_0-C_vV^T\partial_vx_0=B_u(N_0\times V)^T\partial_vx_0+B_v(N_0\times V)^T\partial_ux_0-C_uV^T\partial_vx_0-C_vV^T\partial_ux_0=0$，因此是

$B_u = C_u\frac{V^T\partial_u x_0}{(N_0\times V)^T\partial_u x_0}$，$B_v = C_v\frac{V^T\partial_v x_0}{(N_0\times V)^T\partial_v x_0}$ 和 $(N_0\times V)^T(C_u\partial_v x_0 - C_v\partial_u x_0) = 0$。通过 $V = V_1\partial_u x_0 + V_2\partial_v x_0$ 我们得到 $C_v = -aV_1$，$C_u = aV_2$，$a\subset\mathbb{C}$。

在 (3.2) 的第一个关系中，我们可以用任意等距曲面 x 替换 x_0；因此，$BV + CN_0\times V = bdx_0$，$b\subset\mathbb{C}$ 和 $C = \frac{bdx_0^T(N_0\times V)}{|N_0\times V|^2}$ 等价于之前对 C 的确定。

我们因此得到

$$\frac{\boldsymbol{mn}N_0^T[\partial_w V\times d(V+x_0)]}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2} = \frac{bdx_0^T(N_0\times V)}{|N_0\times V|^2}$$

，并且由于 $\boldsymbol{m}\neq 0$，我们得到

$$\boldsymbol{n}N_0^T[\partial_w V\times d(V+x_0)]\wedge dx_0^T(N_0\times V) = 0 \tag{3.3}$$

和

$$\begin{aligned} b &= \frac{\boldsymbol{mn}|N_0\times V|^2}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)} \\ &= \frac{\boldsymbol{mn}|N_0\times V|^2}{[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]^2}\frac{N_0^T[\partial_w V\times\partial_v(V+x_0)]}{\partial_v x_0^T(N_0\times V)}. \end{aligned} \tag{3.4}$$

现在方程（1.2）变为：

$$\begin{aligned} &\frac{d(V+x_0)^T\odot dN_0}{\boldsymbol{m}}[N_0^T(\partial_w V\times V)+\boldsymbol{n}V^T\partial_w V]-\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0+\boldsymbol{n}V)\odot[-\frac{d\boldsymbol{m}}{\boldsymbol{m}^2}[N_0^T(\partial_w V\times V) \\ &+\boldsymbol{n}V^T\partial_w V]+\frac{1}{\boldsymbol{m}}(dV\times N_0+\boldsymbol{m}dN_0+d\boldsymbol{n}V+\boldsymbol{n}dV)^T\partial_w V+\frac{\partial_w\boldsymbol{m}}{\boldsymbol{m}^2}(V\times N_0+\boldsymbol{n}V)^Td(V+x_0) \\ &-\frac{1}{\boldsymbol{m}}(\partial_w V\times N_0+\partial_w\boldsymbol{n}V+\boldsymbol{n}\partial_w V)^Td(V+x_0)]+\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot \\ &[\frac{d(V+x_0)^T\partial_u N_0}{\boldsymbol{m}}(V\times N_0+\boldsymbol{m}N_0+\boldsymbol{n}V)^T\partial_v(V+x_0)-\frac{d(V+x_0)^T\partial_v N_0}{\boldsymbol{m}}(V\times N_0+\boldsymbol{m}N_0 \\ &+\boldsymbol{n}V)^T\partial_u(V+x_0)-\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0+\boldsymbol{n}V)[-\frac{\partial_u\boldsymbol{m}}{\boldsymbol{m}^2}(V\times N_0+\boldsymbol{n}V)^T\partial_v(V+x_0) \\ &+\frac{\partial_v\boldsymbol{m}}{\boldsymbol{m}^2}(V\times N_0+\boldsymbol{n}V)^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}(\partial_u V\times N_0+V\times\partial_u N_0+\boldsymbol{m}\partial_u N_0+\partial_u\boldsymbol{n}V+\boldsymbol{n}\partial_u V)^T\partial_v(V+x_0) \\ &-\frac{1}{\boldsymbol{m}}(\partial_v V\times N_0+V\times\partial_v N_0+\boldsymbol{m}\partial_v N_0+\partial_v\boldsymbol{n}V+\boldsymbol{n}\partial_v V)^T\partial_u(V+x_0)]]=0. \end{aligned}$$

从 $B = \frac{bdx_0^TV}{|V|^2}$ 中取 $d\boldsymbol{m}$ 并使用 $A=0$，上述方程中 $\boldsymbol{m}$ 和 $\boldsymbol{n}$ 的导数消失，我们只得到：

$$\begin{aligned} &\boldsymbol{n}\{\boldsymbol{n}^2[\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^TV}{\boldsymbol{m}^2}[N_0^T(\partial_w V\times V)N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]- \\ &[(\partial_u V\times V)\times\partial_w V]^T\partial_v(V+x_0)+[(\partial_v V\times V)\times\partial_w V]^T\partial_u(V+x_0)]+ \\ &\frac{1}{\boldsymbol{m}^2}d(V+x_0)^TV\odot dx_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}V^T\partial_w V+ \\ &\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^TV}{\boldsymbol{m}^2}[[\partial_u x_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-\partial_w V^T\partial_u(V+x_0)+ \end{aligned}$$

$$\partial_u V^T \partial_w V]V^T \partial_v(V+x_0) - [\partial_v x_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} - \partial_w V^T \partial_v(V+x_0)+$$

$$\partial_v V^T \partial_w V]V^T \partial_u(V+x_0)]]+\boldsymbol{n}[\frac{N_0^T[\partial_w V \times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]} \odot \frac{d(V+x_0)^T V}{\boldsymbol{m}^2}[|V|^2 V^T \partial_w V N_0^T(\partial_u N_0 \times \partial_v N_0)$$

$$-V^T \partial_w V[N_0^T[\partial_u V \times \partial_v(V+x_0)] - N_0^T[\partial_v V \times \partial_u(V+x_0)]] + \partial_w V^T \partial_u V(N_0 \times V)^T \partial_v(V+x_0)-$$

$$\partial_w V^T \partial_v V(N_0 \times V)^T \partial_u(V+x_0) - [(\partial_u V \times V)\times(N_0 \times \partial_w V)]^T \partial_v(V+x_0) + [(\partial_v V \times V)\times(N_0 \times \partial_w V)]^T \partial_u(V+x_0)]+$$

$$\frac{N_0^T[\partial_w V \times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]} \odot \frac{d(V+x_0)^T(V \times N_0)}{\boldsymbol{m}^2}[N_0^T(\partial_w V \times V)N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]-$$

$$[(\partial_u V \times V)\times \partial_w V]^T \partial_v(V+x_0) + [(\partial_v V \times V)\times\partial_w V]^T \partial_u(V+x_0)]+$$

$$\frac{d(V+x_0)^T \odot dN_0}{\boldsymbol{m}}(\partial_w V^T V)^2 + \frac{1}{\boldsymbol{m}^2} d(V+x_0)^T V \odot dx_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} N_0^T(\partial_w V \times V)+$$

$$\frac{1}{\boldsymbol{m}^2} d(V+x_0)^T(V \times N_0) \odot dx_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} V^T \partial_w V - \frac{1}{\boldsymbol{m}} d(V+x_0)^T V \odot dN_0^T \partial_w V V^T \partial_w V+$$

$$\frac{1}{\boldsymbol{m}^2} d(V+x_0)^T V \odot (\partial_w V \times N_0)^T d(V+x_0) V^T \partial_w V + \frac{N_0^T[\partial_w V \times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]} \odot [$$

$$\frac{d(V+x_0)^T \partial_u N_0}{\boldsymbol{m}} V^T \partial_v(V+x_0) V^T \partial_w V - \frac{d(V+x_0)^T \partial_v N_0}{\boldsymbol{m}} V^T \partial_u(V+x_0) V^T \partial_w V-$$

$$\frac{1}{\boldsymbol{m}} d(V+x_0)^T(V \times N_0)[(-\frac{1}{\boldsymbol{m}} \partial_u x_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} + \frac{1}{\boldsymbol{m}} \partial_w V^T \partial_u(V+x_0)-$$

$$\frac{1}{\boldsymbol{m}} \partial_u V^T \partial_w V) V^T \partial_v(V+x_0) + (\frac{1}{\boldsymbol{m}} \partial_v x_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} - \frac{1}{\boldsymbol{m}} \partial_w V^T \partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}} \partial_v V^T \partial_w V) V^T \partial_u(V+x_0) + \frac{1}{\boldsymbol{m}} \partial_u V^T \partial_v(V+x_0) V^T \partial_w V - \frac{1}{\boldsymbol{m}} \partial_v V^T \partial_u(V+x_0) V^T \partial_w V]-$$

$$\frac{1}{\boldsymbol{m}} d(V+x_0)^T V[(-\frac{1}{\boldsymbol{m}} \partial_u x_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} + \frac{1}{\boldsymbol{m}} \partial_w V^T \partial_u(V+x_0)-$$

$$\frac{1}{\boldsymbol{m}} \partial_u V^T \partial_w V)(V \times N_0)^T \partial_v(V+x_0) + (\frac{1}{\boldsymbol{m}} \partial_v x_0^T V \frac{N_0^T[\partial_w V \times \partial_u(V+x_0)]}{\partial_u x_0^T(N_0 \times V)} - \frac{1}{\boldsymbol{m}} \partial_w V^T \partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}} \partial_v V^T \partial_w V)(V \times N_0)^T \partial_u(V+x_0) - \frac{1}{\boldsymbol{m}} N_0^T(\partial_w V \times \partial_u V) V^T \partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}} N_0^T(\partial_w V \times \partial_v V) V^T \partial_u(V+x_0) + \frac{1}{\boldsymbol{m}} \partial_u V^T \partial_v(V+x_0) N_0^T(\partial_w V \times V)-$$

$$\frac{1}{\boldsymbol{m}} \partial_v V^T \partial_u(V+x_0) N_0^T(\partial_w V \times V) + \frac{1}{\boldsymbol{m}}(\partial_u V \times N_0 + \boldsymbol{m}\partial_u N_0)^T \partial_v(V+x_0) V^T \partial_w V-$$

$$\frac{1}{\boldsymbol{m}}(\partial_v V \times N_0 + \boldsymbol{m}\partial_v N_0)^T \partial_u(V+x_0) V^T \partial_w V + \frac{|V|^2}{\boldsymbol{m}} N_0^T(\partial_u N_0 \times \partial_v N_0) V^T \partial_w V]]]+$$

$$\frac{N_0^T[\partial_w V \times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]} \odot \frac{d(V+x_0)^T V}{\boldsymbol{m}^2}[N_0^T(\partial_w V \times V) N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]+$$

$$|V|^2 N_0^T(\partial_w V \times V) N_0^T(\partial_u N_0 \times \partial_v N_0) - N_0^T(\partial_w V \times V)[N_0^T[\partial_u V \times \partial_v(V+x_0)]-$$

$$N_0^T[\partial_v V \times \partial_u(V+x_0)]] + (N_0 \times \partial_w V)^T \partial_u V(N_0 \times V)^T \partial_v(V+x_0) - (N_0 \times \partial_w V)^T \partial_v V(N_0 \times V)^T \partial_u(V+x_0)+$$

$$\boldsymbol{m}^2 N_0^T(\partial_w V \times V) N_0^T(\partial_u N_0 \times \partial_v N_0)] + \frac{N_0^T[\partial_w V \times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]} \odot \frac{d(V+x_0)^T(V \times N_0)}{\boldsymbol{m}^2}[$$

$$|V|^2 V^T \partial_w V N_0^T(\partial_u N_0 \times \partial_v N_0) - V^T \partial_w V[N_0^T[\partial_u V \times \partial_v(V+x_0)] - N_0^T[\partial_v V \times \partial_u(V+x_0)]]+$$

$$\partial_w V^T \partial_u V(N_0 \times V)^T \partial_v(V+x_0) - \partial_w V^T \partial_v V(N_0 \times V)^T \partial_u(V+x_0)-$$

$$[(\partial_u V \times V)\times(N_0 \times \partial_w V)]^T \partial_v(V+x_0) + [(\partial_v V \times V)\times(N_0 \times \partial_w V)]^T \partial_u(V+x_0)]+$$

$$2\frac{d(V+x_0)^T\odot dN_0}{\boldsymbol{m}}N_0^T(\partial_w V\times V)\partial_w V^TV-\frac{1}{\boldsymbol{m}}d(V+x_0)^TV\odot[dN_0^T\partial_w VN_0^T(\partial_w V\times V)-$$

$$\frac{1}{\boldsymbol{m}}(\partial_w V\times N_0)^Td(V+x_0)N_0^T(\partial_w V\times V)]-\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0)\odot[$$

$$-\frac{1}{\boldsymbol{m}}dx_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}N_0^T(\partial_w V\times V)+dN_0^T\partial_w VV^T\partial_w V-\frac{1}{\boldsymbol{m}}(\partial_w V\times N_0)^Td(V+x_0)V^T\partial_w V]+$$

$$\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot[\frac{d(V+x_0)^T\partial_u N_0}{\boldsymbol{m}}[V^T\partial_v(V+x_0)N_0^T(\partial_w V\times V)+$$

$$(V\times N_0+\boldsymbol{m}N_0)^T\partial_v(V+x_0)V^T\partial_w V]-\frac{d(V+x_0)^T\partial_v N_0}{\boldsymbol{m}}[V^T\partial_u(V+x_0)N_0^T(\partial_w V\times V)+$$

$$(V\times N_0+\boldsymbol{m}N_0)^T\partial_u(V+x_0)V^T\partial_w V]-\frac{1}{\boldsymbol{m}}d(V+x_0)^TV[-\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_u V)(V\times N_0)^T\partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_v V)(V\times N_0)^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}(\partial_u V\times N_0+\boldsymbol{m}\partial_u N_0)^T\partial_v(V+x_0)N_0^T(\partial_w V\times V)-$$

$$\frac{1}{\boldsymbol{m}}(\partial_v V\times N_0+\boldsymbol{m}\partial_v N_0)^T\partial_u(V+x_0)N_0^T(\partial_w V\times V)+\frac{|V|^2}{\boldsymbol{m}}N_0^T(\partial_u N_0\times\partial_v N_0)N_0^T(\partial_w V\times V)]-$$

$$\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0)[(-\frac{1}{\boldsymbol{m}}\partial_u x_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}+\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_u(V+x_0)-$$

$$\frac{1}{\boldsymbol{m}}\partial_u V^T\partial_w V)(V\times N_0)^T\partial_v(V+x_0)-\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_u V)V^T\partial_v(V+x_0)+$$

$$(\frac{1}{\boldsymbol{m}}\partial_v x_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}}\partial_v V^T\partial_w V)(V\times N_0)^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_v V)V^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}\partial_u V^T\partial_v(V+x_0)N_0^T(\partial_w V\times V)+$$

$$\frac{1}{\boldsymbol{m}}(\partial_u V\times N_0+\boldsymbol{m}\partial_u N_0)^T\partial_v(V+x_0)V^T\partial_w V-\frac{1}{\boldsymbol{m}}\partial_v V^T\partial_u(V+x_0)N_0^T(\partial_w V\times V)-$$

$$\frac{1}{\boldsymbol{m}}(\partial_v V\times N_0+\boldsymbol{m}\partial_v N_0)^T\partial_u(V+x_0)V^T\partial_w V+\frac{|V|^2}{\boldsymbol{m}}N_0^T(\partial_u N_0\times\partial_v N_0)V^T\partial_w V]]\}=0.$$

在上述方程中，含有 x_0 的第二基本形式的二次项通过 Gauß 定理变得依赖于 x_0 的度量，并且线性出现的含有 x_0 的第二基本形式的项消失（注意 $N_0^Td(V+x_0)=-dN_0^TV$），我们只得到：

$$\boldsymbol{n}\{\boldsymbol{n}^2[\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^TV}{\boldsymbol{m}^2}[N_0^T(\partial_w V\times V)N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]-$$

$$N_0^T(\partial_w V\times V)N_0^T(\partial_u V\times\partial_v V)+(\partial_w V\times\partial_v x_0)^TN_0N_0^T(V\times\partial_u V)-$$

$$(\partial_w V\times\partial_u x_0)^TN_0N_0^T(V\times\partial_v V)]+\frac{1}{\boldsymbol{m}^2}d(V+x_0)^TV\odot dx_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}V^T\partial_w V+$$

$$\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^TV}{\boldsymbol{m}^2}[[\partial_u x_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-$$

$$\partial_w V^T\partial_u x_0]V^T\partial_v(V+x_0)-[\partial_v x_0^TV\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-\partial_w V^T\partial_v x_0]$$

$$V^T\partial_u(V+x_0)]]+\boldsymbol{n}[\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^TV}{\boldsymbol{m}^2}[|V|^2V^T\partial_w VN_0^T(\partial_u N_0\times\partial_v N_0)$$

$$-V^T\partial_w V[N_0^T[\partial_u V\times\partial_v(V+x_0)]-N_0^T[\partial_v V\times\partial_u(V+x_0)]]+\partial_w V^T\partial_u V(N_0\times V)^T\partial_v(V+x_0)-$$

$$\partial_w V^T\partial_v V(N_0\times V)^T\partial_u(V+x_0)-\partial_u V^T(N_0\times\partial_w V)V^T\partial_v(V+x_0)+\partial_v V^T(N_0\times\partial_w V)V^T\partial_u(V+x_0)+$$

$$N_0^T(\partial_w V\times V)(\partial_u V^T\partial_v x_0-\partial_v V^T\partial_u x_0)]+$$

$$\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^T(V\times N_0)}{\boldsymbol{m}^2}[N_0^T(\partial_w V\times V)N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]-$$

$$\partial_u V^T\partial_w VV^T\partial_v(V+x_0)+\partial_v V^T\partial_w VV^T\partial_u(V+x_0)+V^T\partial_w V(\partial_u V^T\partial_v x_0-\partial_v V^T\partial_u x_0)]+$$

$$\frac{1}{\boldsymbol{m}^2}d(V+x_0)^T V\odot dx_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}N_0^T(\partial_w V\times V)+$$

$$\frac{1}{\boldsymbol{m}^2}d(V+x_0)^T(V\times N_0)\odot dx_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}V^T\partial_w V+$$

$$\frac{1}{\boldsymbol{m}^2}d(V+x_0)^T V\odot(\partial_w V\times N_0)^T d(V+x_0)V^T\partial_w V+\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot[-$$

$$\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0)[(-\frac{1}{\boldsymbol{m}}\partial_u x_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}+\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_u x_0)V^T\partial_v(V+x_0)$$

$$+(\frac{1}{\boldsymbol{m}}\partial_v x_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_v x_0)V^T\partial_u(V+x_0)$$

$$+\frac{1}{\boldsymbol{m}}V^T\partial_w V(\partial_u V^T\partial_v x_0-\partial_v V^T\partial_u x_0)]-$$

$$\frac{1}{\boldsymbol{m}}d(V+x_0)^T V[(-\frac{1}{\boldsymbol{m}}\partial_u x_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}+\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_u x_0)(V\times N_0)^T\partial_v(V+x_0)+$$

$$(\frac{1}{\boldsymbol{m}}\partial_v x_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_v x_0)(V\times N_0)^T\partial_u(V+x_0)-$$

$$\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_u V)V^T\partial_v(V+x_0)+\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_v V)V^T\partial_u(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times V)(\partial_u V^T\partial_v x_0-\partial_v V^T\partial_u x_0)+\frac{1}{\boldsymbol{m}}(\partial_u V\times N_0)^T\partial_v(V+x_0)V^T\partial_w V-$$

$$\frac{1}{\boldsymbol{m}}(\partial_v V\times N_0)^T\partial_u(V+x_0)V^T\partial_w V+\frac{|V|^2}{\boldsymbol{m}}N_0^T(\partial_u N_0\times\partial_v N_0)V^T\partial_w V]]]+$$

$$\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^T V}{\boldsymbol{m}^2}[N_0^T(\partial_w V\times V)N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]+$$

$$|V|^2N_0^T(\partial_w V\times V)N_0^T(\partial_u N_0\times\partial_v N_0)-N_0^T(\partial_w V\times V)[N_0^T[\partial_u V\times\partial_v(V+x_0)]-$$

$$N_0^T[\partial_v V\times\partial_u(V+x_0)]]+(N_0\times\partial_w V)^T\partial_u V(N_0\times V)^T\partial_v(V+x_0)-(N_0\times\partial_w V)^T\partial_v V(N_0\times V)^T\partial_u(V+x_0)+$$

$$\boldsymbol{m}^2N_0^T(\partial_w V\times V)N_0^T(\partial_u N_0\times\partial_v N_0)]+\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot\frac{d(V+x_0)^T(V\times N_0)}{\boldsymbol{m}^2}[$$

$$|V|^2V^T\partial_w VN_0^T(\partial_u N_0\times\partial_v N_0)-V^T\partial_w V[N_0^T[\partial_u V\times\partial_v(V+x_0)]-N_0^T[\partial_v V\times\partial_u(V+x_0)]]+$$

$$\partial_w V^T\partial_u V(N_0\times V)^T\partial_v(V+x_0)-\partial_w V^T\partial_v V(N_0\times V)^T\partial_u(V+x_0)-$$

$$\partial_u V^T(N_0\times\partial_w V)V^T\partial_v(V+x_0)+\partial_v V^T(N_0\times\partial_w V)V^T\partial_u(V+x_0)+N_0^T(\partial_w V\times V)(\partial_u V^T\partial_v x_0-\partial_v V^T\partial_u x_0)]+$$

$$\frac{1}{\boldsymbol{m}^2}d(V+x_0)^T V\odot(\partial_w V\times N_0)^T d(V+x_0)N_0^T(\partial_w V\times V)-\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0)\odot[$$

$$-\frac{1}{\boldsymbol{m}}dx_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}N_0^T(\partial_w V\times V)-\frac{1}{\boldsymbol{m}}(\partial_w V\times N_0)^T d(V+x_0)V^T\partial_w V]+$$

$$\frac{N_0^T[\partial_w V\times d(V+x_0)]}{N_0^T[\partial_u(V+x_0)\times\partial_v(V+x_0)]}\odot[d(V+x_0)^T(N_0\times V)V^T\partial_w VN_0^T(\partial_u N_0\times\partial_v N_0)-$$

$$\frac{1}{\boldsymbol{m}}d(V+x_0)^T V[-\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_u V)(V\times N_0)^T\partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_v V)(V\times N_0)^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}(\partial_u V\times N_0)^T\partial_v(V+x_0)N_0^T(\partial_w V\times V)-$$

$$\frac{1}{\boldsymbol{m}}(\partial_v V\times N_0)^T\partial_u(V+x_0)N_0^T(\partial_w V\times V)+\frac{|V|^2}{\boldsymbol{m}}N_0^T(\partial_u N_0\times\partial_v N_0)N_0^T(\partial_w V\times V)]-$$

$$\frac{1}{\boldsymbol{m}}d(V+x_0)^T(V\times N_0)[(-\frac{1}{\boldsymbol{m}}\partial_u x_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}+\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_u(V+x_0)-$$

$$\frac{1}{\boldsymbol{m}}\partial_u V^T\partial_w V)(V\times N_0)^T\partial_v(V+x_0)-\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_u V)V^T\partial_v(V+x_0)+$$

$$(\frac{1}{\boldsymbol{m}}\partial_v x_0^T V\frac{N_0^T[\partial_w V\times\partial_u(V+x_0)]}{\partial_u x_0^T(N_0\times V)}-\frac{1}{\boldsymbol{m}}\partial_w V^T\partial_v(V+x_0)+$$

$$\frac{1}{\boldsymbol{m}}\partial_v V^T\partial_w V)(V\times N_0)^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}N_0^T(\partial_w V\times\partial_v V)V^T\partial_u(V+x_0)+\frac{1}{\boldsymbol{m}}\partial_u V^T\partial_v x_0 N_0^T(\partial_w V\times V)+$$

$$\frac{1}{\boldsymbol{m}}(\partial_u V\times N_0)^T\partial_v(V+x_0)V^T\partial_w V-\frac{1}{\boldsymbol{m}}\partial_v V^T\partial_u x_0 N_0^T(\partial_w V\times V)-$$

$$\frac{1}{\boldsymbol{m}}(\partial_v V\times N_0)^T\partial_u(V+x_0)V^T\partial_w V+\frac{|V|^2}{\boldsymbol{m}}N_0^T(\partial_u N_0\times\partial_v N_0)V^T\partial_w V]]\}=0.$$

3.2. **情况** $m=V+\boldsymbol{m}N_0$.

我们有 $0=(Rm)^T\tilde{d}(RV+x)=m^T(\omega\times V+d(V+x_0)+\partial_w V dw)$，或者，假设 $m^T\partial_w V\neq 0$，

$$dw=\frac{\boldsymbol{m}V^T(\omega\times N_0+dN_0)-V^T d(V+x_0)}{V^T\partial_w V}.$$

应用兼容条件 d 到此方程并使用该方程本身我们得到

$$0=-\frac{\partial_w[\boldsymbol{m}V^T(\omega\times N_0+dN_0)-V^T d(V+x_0)]}{V^T\partial_w V}\wedge\frac{\boldsymbol{m}V^T(\omega\times N_0+dN_0)-V^T d(V+x_0)}{V^T\partial_w V}+$$

$$d\frac{\boldsymbol{m}V^T}{V^T\partial_w V}\wedge(\omega\times N_0+dN_0)-d\frac{V^T}{V^T\partial_w V}\wedge d(V+x_0)\overset{(2.1)}{=}$$

$$-\frac{\partial_w[V^T d(V+x_0)]}{V^T\partial_w V}\wedge\frac{V^T d(V+x_0)}{V^T\partial_w V}-d\frac{V^T}{V^T\partial_w V}\wedge d(V+x_0)-\frac{\boldsymbol{m}^2 N_0^T(\partial_w V\times V)N_0^T(dN_0\times\wedge dN_0)}{2(V^T\partial_w V)^2}$$

$$+[-\frac{\partial_w\boldsymbol{m}V^T d(V+x_0)}{(V^T\partial_w V)^2}+\frac{\boldsymbol{m}\partial_w[V^T d(V+x_0)]}{(V^T\partial_w V)^2}+d\frac{\boldsymbol{m}}{V^T\partial_w V}]\wedge V^T(\omega\times N_0+dN_0)$$

$$+\frac{\boldsymbol{m}N_0^T[\partial_w V\times d(V+x_0)]}{(V^T\partial_w V)^2}\wedge(N_0\times V)^T(\omega\times N_0+dN_0).$$

如同前面的情况一样我们得到

$$\frac{\boldsymbol{m}N_0^T[\partial_w V\times d(V+x_0)]}{(V^T\partial_w V)^2}=\frac{b\,dx_0^T(N_0\times V)}{|N_0\times V|^2}$$

使用 $b\subset\mathbb{C}$。

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